Lesson 2-1 Rates of Change & Limits

2016

AP Calculus AB
Lesson 2-1: *Rates of Change & Limits*

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Date

Learning Goals:

- *I can calculate average and instantaneous speeds.*
- *I can define and calculate limits for function values and apply the properties of limits.*
- *I can use the Sandwich Theorem to find certain limits indirectly.*

I. Math 2 Flashback!

Each year physics students at UCSD Muir College drop a gigantic pumpkin from the roof of one of its dorms. (Evidently physics students all across our country are fascinated with dropping pumpkins from high places!) Dense objects such as pumpkins dropped from rest will fall $y = 16t^2$ feet in the first t seconds.
https://www.youtube.com/watch?v=0u1-kNLuA7s

1. What is the average speed of pumpkin during the 3 seconds of the fall? $t_1 = 0$ $t_2 = 3$

$$\frac{\Delta y}{\Delta t} = \frac{f(3) - f(0)}{3 - 0} = \frac{144}{3} = \boxed{48 \text{ ft/sec}}$$

2. Find the speed of the pumpkin at the instant it hits the ground.

$t_1 = 3$ $t_2 = 3$

$\Delta t = 3 - 3 = 0$ ← can't ÷ by zero

$t_1 = 3$ $t_2 = 3 + h$

some time right before 3; e.g. $\underbrace{2.9, 2.99, 2.999}_{\text{Limit.....}}$

$$\frac{dy}{dt} = \lim_{h \to 0} \frac{f(3+h) - f(3)}{(3+h) - 3}$$
$$= \lim_{h \to 0} \frac{16(3+h)^2 - 144}{h}$$
$$= \lim_{h \to 0} \frac{16(9 + 6h + h^2) - 144}{h}$$
$$= \lim_{h \to 0} \frac{144 + 96h + 16h^2 - 144}{h}$$
$$= \lim_{h \to 0} \frac{\cancel{h}(96 + 16h)}{\cancel{h}} = \lim_{h \to 0} 96 + 16h = \boxed{96 \text{ ft/sec}}$$

II. Evaluating Limits Algebraically

Many limits can be evaluated by substitution (and some algebra). Check your answer graphically.

3. Find $\lim_{x \to 2} \frac{x^2 - 4}{x - 2}$

$$\lim_{x \to 2} \frac{(x+2)\cancel{(x-2)}}{\cancel{x-2}}$$

$$\lim_{x \to 2} x + 2$$

$$2 + 2 = \boxed{4}$$

4. Find $\lim_{x \to -3} 2x^3 - 4x^2$

$$\lim_{x \to -3} 2(-3)^3 - 4(-3)^2$$

$$= \boxed{-90}$$

OVER →

Theorem: Given real numbers C and L and $f(x)$ some real-valued function, we say

$$\lim_{x\to C} f(x) = L$$

if and only if

$$\lim_{x\to C^-} f(x) = \lim_{x\to C^+} f(x) = L \qquad LsL = RsL$$

- This means for a limit to exist, the limit from the left $(x \to C^-)$ must equal the limit from the right $(x \to C^+)$.

If you check this condition against problems 3 and 4, you will see that it is true.

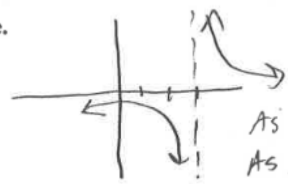
Evaluate the following limits either graphically or by using a table.

Must be memorized (All to TNBM)

5.** $\displaystyle \lim_{x\to 0} \frac{\sin x}{x} = 1$

-.1	.998
-.01	.999
-.001	.9999
.001	.9999
.01	.999
.1	.998

6. $\displaystyle \lim_{x\to 3} \frac{x+3}{x-3}$



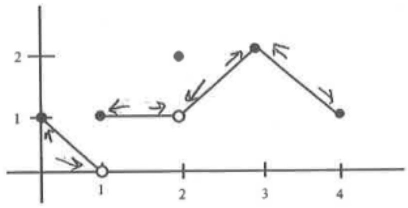
As $x \to 3^-$, $f(x) \to -\infty$
As $x \to 3^+$, $f(x) \to \infty$

Since $LsL \neq RsL$ The limit DNE.

III. Exploring Right- and Left-Handed Limits

7. Consider the following piece-wise function and its graph:

$$f(x) = \begin{cases} -x+1, & 0 \le x < 1 \\ 1, & 1 \le x < 2 \\ 2, & x = 2 \\ x-1, & 2 < x \le 3 \\ -x+5, & 3 < x \le 4 \end{cases}$$



All of the following statements about the function $y = f(x)$ are true. Explain why each is true.

At $x = 0$: $\lim_{x\to 0^+} f(x) = 1$ As x approaches 0 from the right, f(x) approaches 1

At $x = 1$: $\lim_{x\to 1^-} f(x) = 0$ even though $f(1) = 1$ As $x \to 1^-$ (1 from the left), f(x) approaches 0 (point doesn't matter!)

$\lim_{x\to 1^+} f(x) = 1$ As $x \to 1^+$, f(x) approaches

$\lim_{x\to 1} f(x)$ does not exist $\lim_{x\to 1^-} f(x) \neq \lim_{x\to 1^+} f(x)$

At $x = 2$: $\lim_{x\to 2^-} f(x) = 1$ (even though $f(2) = 2$) y-values are approaching 1

$\lim_{x\to 2^+} f(x) = 1$ (even though $f(2) = 2$) y-values are approaching 1

$\lim_{x\to 2} f(x) = 1$ even though $f(2) = 2$ $LsL = 1 = RsL$

At $x = 3$: $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x) = 2 = \lim_{x\to 3} f(x)$ $LsL = 2 = RsL$

At $x = 4$: $\lim_{x\to 4^-} f(x) = 1$ ☆ $\lim_{x\to 4^+}$ DNE b/c the function DNE

At noninteger values of c between 0 and 4, f has a limit as $x \to c$

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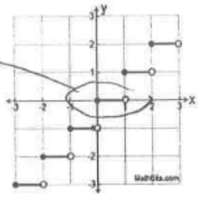
8. The **greatest integer function**, $f(x)=\lfloor x \rfloor$ or $f(x)=\text{int}\,x$ is a piecewise function that is defined to be *the greatest integer* $\le x$. So $f(x)=\lfloor 6.8 \rfloor = \text{int}\,6.8 = 6$. The graph of the greatest integer function is pictured. Given $c \in \mathbb{Z}$, explain why $\lim_{x\to c}(\text{int}\,x)$ does not exist.



$\lim_{x\to 0^-} \text{int}\,x = -1$, but $\lim_{x\to 0^+} \text{int}\,x = 0$

$LsL = -1 \neq RsL = 0$

IV. <u>Theorem:</u> Properties of Limits

If L, , c, and k are real numbers, and $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$, then ,

Sum Rule: $\lim_{x\to c}[f(x)+g(x)] = L+M$ "Limit of a sum is the sum of the limits."

Difference Rule: $\lim_{x\to c}[f(x)-g(x)] = L-M$

Product Rule: $\lim_{x\to c}[f(x)\cdot g(x)] = L\cdot M$

Constant Multiple Rule $\lim_{x\to c}[k\cdot f(x)] = k\cdot L$

Quotient Rule: $\lim_{x\to c}\dfrac{f(x)}{g(x)} = \dfrac{L}{M}$, $M \neq 0$

Power Rule: $\lim_{x\to c}(f(x))^{r/s} = L^{r/s}$, provided $L^{r/s}$ is $\in \mathbb{R}$

***Note that without knowing it, we have technically used a few of these properties in problems 3 and 4.

OVER →

Find the following limits using the properties of limits (and algebra). **Confirm your answers graphically**

9. Determine $\lim_{x\to 0}\frac{\tan x}{x}$

$= \lim_{x\to 0} \frac{\frac{\sin x}{\cos x}}{x}$

$= \lim_{x\to 0} \frac{\sin x}{x \cdot \cos x}$

$= \lim_{x\to 0} \frac{\sin x}{x} \cdot \frac{1}{\cos x} = 1 \cdot \frac{1}{1} = \boxed{1}$

10. Determine $\lim_{x\to 0}\frac{x+\sin x}{x}$

$= \lim_{x\to 0} \left[\frac{x}{x} + \frac{\sin x}{x}\right]$

$= \lim_{x\to 0} [1+1]$

$= \boxed{2}$

11. Find $\lim_{x\to 0}\frac{\sin^2 x}{x}$

$= \lim_{x\to 0} \frac{\sin x \cdot \sin x}{x}$

$= \lim_{x\to 0} = \frac{\sin x}{x} \cdot \sin x = 1 \cdot 0 = \boxed{0}$

12. If $\lim_{x\to c} f(x)=2$ and $\lim_{x\to c} g(x)=8$, find:

a. $\lim_{x\to c}[f(x)+g(x)]$

Sum Rule

$2+8=\boxed{10}$

b. $\lim_{x\to c}[f(x)g(x)]$

Product Rule

$2 \cdot 8 = \boxed{16}$

c. $\lim_{x\to c}\frac{f(x)}{g(x)}$

Quotient Rule

$\frac{2}{8} = \boxed{\frac{1}{4}}$

d. $\lim_{x\to c} 3\cdot g(x) = 3 \cdot \lim_{x\to c} g(x) = 3 \cdot 8$

Constant Multiple Rule

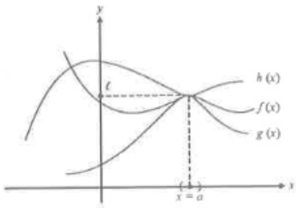
$= \boxed{24}$

e. $\lim_{x\to c}(f(x)-g(x))^2$

$(2-8)^2 = (-6)^2 = \boxed{36}$

V. <u>Theorem:</u> Sandwich Theorem

If $g(x)\le f(x)\le h(x)$ for all $x \ne a$ in some interval about a, and $\lim_{x\to a} g(x)=\lim_{x\to a} h(x)=L$, then $\lim_{x\to a} f(x)=L$.



☆ This will be used to verify $\lim_{x\to 0}\frac{\sin x}{x}=1$

VI. Practice (NO CALCULATOR):

1. $\lim_{x \to 0} \frac{\sin x}{x^2 + 3x}$ is (Sample AP Exam problem)

(A) 1 (B) $\frac{1}{3}$ (C) 3 (D) ∞

$$\lim_{x \to 0} \frac{\sin x}{x(x+3)} = \lim_{x \to 0} \left[\frac{\sin x}{x} \cdot \frac{1}{x+3}\right] = 1 \cdot \frac{1}{3}$$

Evaluate each of the following limits without the aid of a calculator.

2. $\lim_{x \to 2} 4x$

$= 4 \cdot 2$

$= \boxed{8}$

3. $\lim_{y \to -1} 3y^4 - 6y^3 - 2y$

$= 3(-1)^4 - 6(-1)^3 - 2(-1)$

$= 3 + 6 + 2$

$= \boxed{11}$

4. $\lim_{x \to 1} \frac{2x-2}{x-1}$

$= \lim_{x \to 1} \frac{2\cancel{(x-1)}}{\cancel{x-1}}$

$= \boxed{2}$

5. $\lim_{x \to 2} \frac{x^2 - 4x + 4}{x^2 + x - 6}$

$= \lim_{x \to 2} \frac{(x-2)\cancel{(x-2)}}{(x+3)\cancel{(x-2)}}$

$= \frac{2-2}{5} = \boxed{0}$

6. $\lim_{x \to 3} \frac{x}{x-3}$

$x \to 3^- \Rightarrow -\infty$

$x \to 3^+ \Rightarrow +\infty$

$\boxed{DNE}$

7. $\lim_{x \to 4} \frac{3-x}{x^2 - 2x - 8}$

$\lim_{x \to 4} \frac{-(x-3)}{(x-4)(x+2)}$

$x \to 4^- \Rightarrow \frac{-}{- \cdot +} \to \infty$

$x \to 4^+ \Rightarrow \frac{-}{+ \cdot +} \to -\infty$

$\boxed{\lim_{x \to 4} \text{DNE}}$

8. $\lim_{x \to 3} \frac{-x^2}{x^2 - 6x + 9}$

$\lim_{x \to 3} \frac{-x^2}{(x-3)(x-3)}$

$x \to 3^- \Rightarrow \frac{-}{- \; -} \to -\infty$

$x \to 3^+ \Rightarrow \frac{-}{+ \; +} \to -\infty$

$\boxed{\lim_{x \to 3} \to -\infty}$ or DNE

9. $f(x) = \begin{cases} \frac{x-2}{x-1}, x \geq 1 \\ \frac{x}{x-1}, x < 1 \end{cases}$ find $\lim_{x \to 1} f(x)$.

$\lim_{x \to 1^+} \frac{x-2}{x-1} \Rightarrow \frac{-}{+} \to -\infty$

$\lim_{x \to 1^-} \frac{x}{x-1} \Rightarrow \frac{+}{-} \to -\infty$

$\boxed{-\infty \text{ or DNE}}$

10. $f(x) = \begin{cases} \frac{x^2 - 49}{x-7}, x \neq 7 \\ k^2 - 2, x = 7 \end{cases}$ find k such that $\lim_{x \to 7} f(x) = f(7)$

$\frac{x^2 - 49}{x-7} = \frac{(x+7)\cancel{(x-7)}}{\cancel{x-7}} = x + 7 \Rightarrow \lim_{x \to 7} x + 7 = 14$

$f(7)$ must equal 14.

$14 = k^2 - 2$

$16 = k^2$

$\boxed{k = \pm 4}$